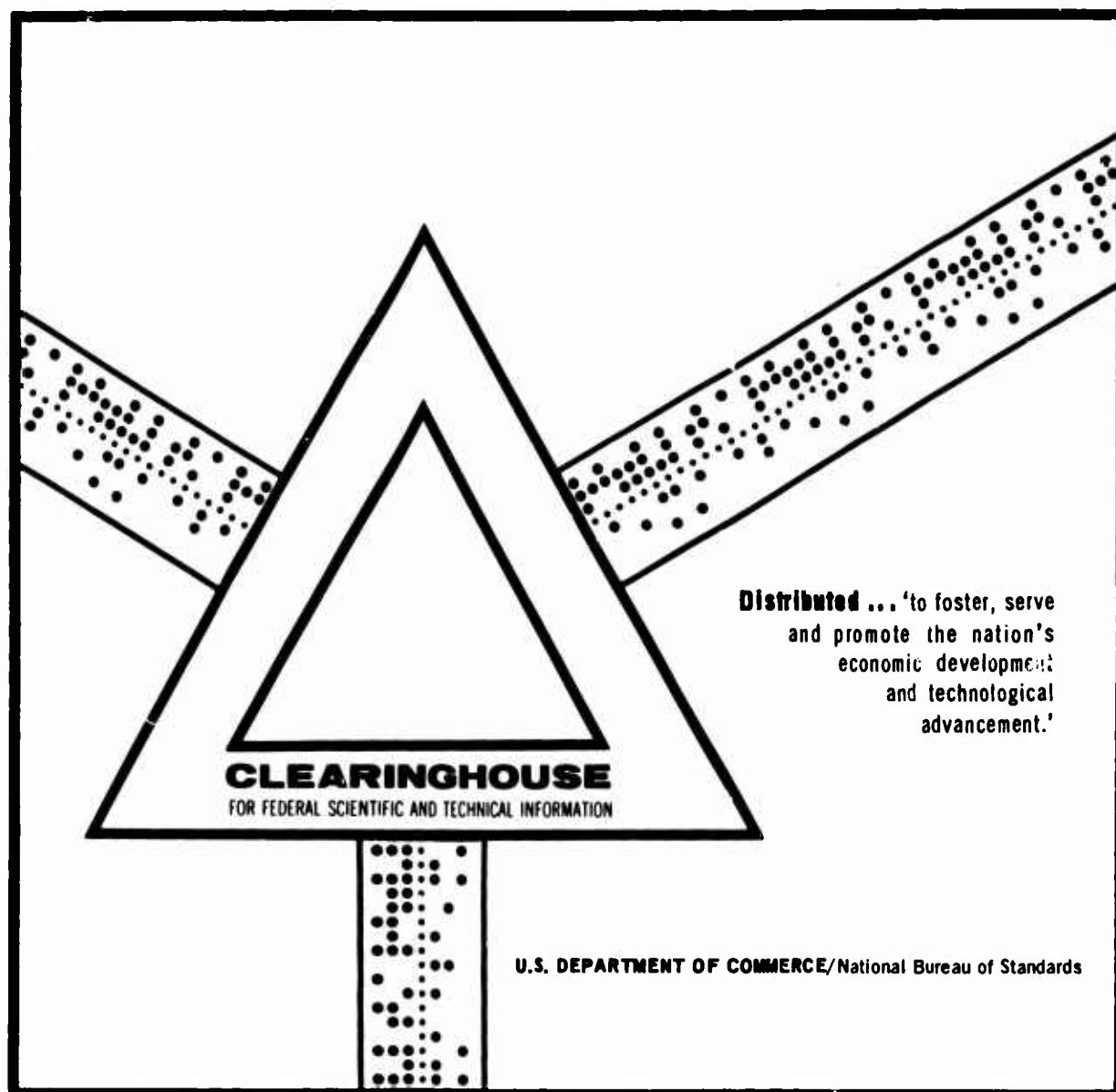AD 701 888

RENEWAL REWARD PROCESSES

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September 1969

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by
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ABSTRACT

We considers a process in which rewards are being earned
and for which there exist time points at which the process
begins anew. That is, ~~we suppose~~ that there exists an
embedded renewal process. An expression for the
asymptotic mean reward earned during any time interval is
then obtained. In the final section ~~we consider~~ the special
case of a regenerative reward process and ~~we present~~ a
simple expression for the long run average reward earned
per unit time.

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0. INTRODUCTION

Let $X_1, X_2, \ldots$, be the interarrival times for a renewal process with interarrival distribution F . Suppose that at the time of the ith renewal we receive a reward Y_i . Y_i may depend on X_i , but it is assumed that the pairs (X_i, Y_i) , $i = 1, 2, \ldots$, are independent and identically distributed. If we let

$$Y(t) = \sum_{i=1}^{N(t)} Y_i ,$$

where $N(t)$ is the number of renewals by time t , then $Y(t)$ represents the total reward earned by time t . The stochastic process $\{Y(t), t \geq 0\}$ is called a renewal reward process. In the first section of this paper, we will prove the analogue of Blackwell's theorem for a renewal reward process. In Section 3 we will consider processes of the form $Y(t) = \int_0^t V(s)ds$, where V is a real-valued regenerative process. An important result of Smith [5], p. 262, asserts under mild conditions that $\frac{Y(t)}{t}$ converges a.s. and in expectation to κ_1/μ_1 , where κ_1 is the expected value of the integral of V over a regenerative cycle and μ_1 is the expected length of the regeneration cycle. Our result is that $\kappa_1/\mu_1 = EV(\infty)$, where $V(\infty)$ is the limiting distribution of $V(t)$.

1. BLACKWELL'S THEOREM FOR RENEWAL REWARD PROCESSES

The following proposition is well known:

Proposition 1:

If either EY_1 or EX_1 is finite, then

(i) $\lim\limits_{t\to\infty} \dfrac{Y(t)}{t} = \dfrac{EY_1}{EX_1}$ with probability 1, and

(ii) $\lim\limits_{t\to\infty} \dfrac{EY(t)}{t} = \dfrac{EY_1}{EX_1}$.

A proof, based on a Tauberian theorem, is given by Johns and Miller [2] and credited to Bell. The proposition also follows from a more general result of Smith [5]. Part (i) of the above is clearly the analogue of the elementary renewal theorem. We shall now prove the analogue of Blackwell's theorem.

Theorem 1:

If $EY_1 < \infty$, F is not lattice, and $EX_1 Y_1 < \infty$, then

$$\lim\limits_{t\to\infty} E[Y(t + h) - Y(t)] = h \frac{EY_1}{EX_1} \quad \text{for all} \quad h \geq 0 .$$

Proof:

Let $m(t) = E[N(t)]$. Now,

$$E[Y(t)] = E\left[\sum_{i=1}^{N(t)+1} Y_i\right] - E[Y_{N(t)+1}]$$

(1)

$$= (m(t) + 1)EY_1 - E[Y_{N(t)+1}]$$

where the last identity follows from Wald's equation. Hence,

$$E[Y(t + h) - Y(t)] = (m(t + h) - m(t))EY_1 - E[Y_{N(t+h)+1} - Y_{N(t)+1}] ,$$

and the result would follow from Blackwell's theorem if we can show that $\lim_{t \to \infty} E[Y_{N(t)+1}]$ exists and is finite. Toward this end, let $g(t) = E[Y_{N(t)+1}]$. Then

$$g(t) = \int_0^\infty E[Y_{N(t)+1} \mid X_1 = x] dF(x)$$

$$\int_t^\infty E[Y_1 \mid X_1 = x] dF(x) + \int_0^t g(t - x) dF(x) .$$

This renewal type equation has the solution

$$(2) \qquad g(t) = h(t) + \int_0^t h(t - x) dm(x) ,$$

where

$$h(t) = \int_t^\infty E[Y_1 \mid X_1 = x] dF(x) .$$

Suppose now that all rewards are nonnegative; then by the key renewal theorem, it follows that

$$\lim_{t \to \infty} g(t) = \frac{\int_0^\infty h(t) dt}{EX_1} = \frac{\int_0^\infty \int_t^\infty F[Y_1 \mid X_1 = x] dF(x) dt}{EX_1}$$

$$= \frac{\int_0^\infty x E[Y_1 \mid X_1 = x] dF(x)}{EX_1} = \frac{EX_1 Y_1}{EX_1}$$

where the interchange of integrals is justified by the nonnegativity of rewards. In the general case, the result may be proven by breaking up the rewards into their positive and negative parts and applying the above argument separately to each.

Remark 1:

The proof of Theorem 1 may also be used to prove (ii) of Proposition 1. This is done in the following manner: Assume first that $EY_1 < \infty$. Then, from Equation (1) and the elementary renewal theorem, it follows that (ii) holds if $\frac{g(t)}{t} \to 0$. This, however, easily follows from (2), the assumption that $EY_1 < \infty$, and the elementary renewal theorem. If $EY_1 = \infty$ (but $EX_1 < \infty$) , then the result follows by truncation.

In the above, we have assumed that the rewards are earned at the end of the renewal intervals. However, in many applications the rewards (or costs) are earned gradually during the renewal intervals. For instance, in an inventory model for which an (s,S) policy is employed, the costs are gradually incurred during the renewal cycle. In order to generalize Theorem 1 to include this possibility, let $W(s)$ denote the expected reward earned during the first s time units of a renewal interval of length greater than s . Then, the expected reward earned by t , $EY(t)$ will be given by

$$E[Y(t)] = E\left[\sum_{i=1}^{N(t)} Y_i\right] + E[W(Z(t))]$$

where

$$Z(t) = t - \sum_{i=1}^{N(t)} X_i$$

is the age of the renewal process at time t. Let

$$F_t(a) = \begin{cases} F(a) - \int_0^{t-a} (1 - F(t - y))dm(y) , & a \le t \\ \\ 1 , & a > t . \end{cases}$$

It is well known that F_t is the distribution of Z_t.

Following Smith [6], p. 11, define G to be the class of all distributions F on $[0,\infty)$ having the property that for some K, the Kth iterated convolution of F with itself has an absolutely continuous component.

Theorem 2:

(i) If F is nonlattice, $EY_1 < \infty$, $EX_1 < \infty$, $EX_1Y_1 < \infty$, then if W is continuous and uniformly integrable with respect to the family $\{F_t, t > 0\}$, then

(3)
$$\lim_{t \to \infty} E[Y(t + h) - Y(t)] = h \frac{EY_1}{EX_1}$$

(ii) Under the conditions of (i) but with $F \in G$, (3) holds iff W is uniformly integrable with respect to $\{F_t, t > 0\}$ (W need not be continuous).

Proof:

(i) It follows from Smith [5], p. 259 condition B, that $Z(t)$ converges in distribution to a random variable with c.d.f.

$$F_e(a) = \frac{\int_0^a (1 - F(x))dx}{EX_1} .$$

Thus $E(W(Z(t)))$ converges to $\int_0^\infty W(x)dF_e(x)$ by the Helly-Bray theory

(Loéve [3], p. 183). Hence, the result follows from Theorem 1.

(ii) It follows from Smith [5], p. 259 condition C, that $P(Z_t \in A)$

converges for all Borel sets A to $F_e(A) = \int_A (1 - F(x))dx/EX_1$.

$$\text{Let } W^a(x) = \begin{cases} W(x), & |W(x)| \leq a \\ 0 & |W(x)| > a. \end{cases}$$

Let $W^{a,\delta}$ be a simple function having the property that $\sup_x |W^a(x) - W^{a,\delta}(x)| < \delta$.

$W^{a,\delta}$ can be chosen by choosing $\frac{1}{n} < \delta$ and letting $W^{a,\delta}(x) = \frac{i}{n}$ when

$\frac{i}{n} \leq W(x) < \frac{i+1}{n}$, $i = -\frac{a}{n} \ldots \frac{a}{n}$. Note that the strong convergence in distribution

implies that $E_{F_t} W^{a,\delta} \to E_{F_e} W^{a,\delta}$ for all a, δ. The result now follows by:

$$|E_{F_e} W - E_{F_t} W| \leq |E_{F_e} W - E_{F_e} W^a| + |E_{F_e} W^a - E_{F_e} W^{a,\delta}| + |E_{F_e} W^{a,\delta} - E_{F_t} W^{a,\delta}|$$

$$+ |E_{F_t} W^{a,\delta} - E_{F_t} W^a| + |E_{F_t} W^a - E_{F_t} W|.$$

Now the 1st and 5th terms on the right go to 0 uniformly in t as $a \to \infty$, by

assumption. The 2nd and 4th terms on the right go to 0 uniformly in t, for

fixed a, as $\delta \to 0$. The 3rd term goes to 0 as $t \to \infty$ for fixed a, δ.

Thus, by first choosing a sufficiently large, then fixing a and choosing δ

sufficiently small, and then fixing a, δ and choosing t sufficiently large,

we can make the right-side smaller than any preassigned $\epsilon > 0$.

The necessity of uniform integrability follows from an argument in Loéve [3],
p. 183.

2. REGENERATIVE REWARD PROCESSES

Let $\{V(t), t \geq 0\}$ be a regenerative process [5], p. 256, with imbedded generalized renewal sequence $\{X_i, i \geq 0\}$. By generalized renewal sequence we mean that X_o is independent of the i.i.d. sequence $\{X_i, i > 0\}$ but may have a different distribution. The random elements $V(t)$ take values in an abstract measurable space (F, A). If F, the distribution of X_1, belongs to G and if $\mu_1 = EX_1 < \infty$, then it follows from Smith [5], p. 259, that:

$$
(4) \qquad Pr(X_t \in A) = \frac{1}{\mu_1} \int_0^{\infty} Pr(V_t \in A, X_o > t \mid \text{renewal at } 0)dt
$$

$$
= \mu_\infty(A) \quad \text{for all } A \in A .
$$

It follows from Fubini's theorem that μ_∞ is a probability measure on (F, A). If in addition V is a real valued process with a measurable modification then it follows from Smith [5], p. 262, that $\frac{1}{t} \int_0^t V(s)ds$ converges a.s. and in

expectation to κ_1/μ_1, where $\kappa_1 = E \int_{X_o}^{X_o + X_1} V(s)ds$ (assuming κ_1 exists). A

natural question to pose is whether or not $\kappa_1/\mu_1 = E(V(\infty)) = \int_{-\infty}^{\infty} x \, d\mu_\infty(x)$. We will show that this is the case.

It will be convenient to convert the imbedded renewal process $\{X_i, i \geq 0\}$ into a stationary renewal process. This can be done by inserting a renewal to the left of 0, its distance from 0 having the same distribution as the limiting distribution of $Z(t)$, the age of the renewal process at time t, discussed in Section 1. Formally, we let $\{X_i', i = 0, \pm 1, \ldots\}$ be a doubly infinite sequence of i.i.d. random variables distributed as X_1. Let $T_n = X_o' + \sum_1^n X_i'$ for $n \geq 0$, $T_n = X_o' - \sum_{i=1}^{-n} X_{-i}'$ for $n < 0$. Then $\{T_n, n = 0, \pm 1, \ldots\}$ generates a strictly stationary renewal process on $(-\infty, \infty)$ (see [1], p. 162).

Start the regenerative process V at the first T_1 point to the left of 0. Call this point T^o and call the resulting regenerative process V'. Now

$$\Pr(V'_o \in A) = E\left[\Pr\left(V'_o \in A \mid T^o\right)\right]$$

(5)
$$= \frac{1}{\mu_1} \int_{t=0}^{\infty} \Pr\left(V'_o \in A \mid T^o = -t\right) 1 - F(t) dt .$$

But

(6) $\Pr(V_t \in A, X_o > t \mid \text{renewal at } 0) = \Pr(V_t \in A \mid X_o > t, \text{renewal at } 0)$.

$$\Pr(X_o > t \mid \text{renewal at } 0) = (1 - F(t))\Pr(V_t \in A \mid Z(t) = t)$$

$$= (1 - F(t))\Pr\left(V'_o \in A \mid T^o = -t\right) .$$

Thus, from (4), (5), (6)

(7) $$\Pr(V'_o \in A) = \Pr(V(\infty) \in A) = \mu_\infty(A) .$$

Assume that V has a measurable modification and that $E|V'_o| < \infty$. This implies that $Y(t) = \int_0^t V'(s) ds$ exists a.s. for all t. Start V with a renewal at time 0 (thus X_o has same distribution as X_1) and call the resulting process V''. Define:

$$W(t) = \begin{cases} V''(t) , & t < X_o \\ 0 & t \geq X_o . \end{cases}$$

Then $\int_0^{X_o} |V''(s)| ds = \int_0^{\infty} |W(s)| ds$, possibly infinite. Now

$$\int_0^\infty E|W(s)|\,ds = \int_0^\infty E(|V''(s)| \mid X_o > s)(1 - F(s))\,ds$$

$$= \int_0^\infty E(|V(s)| \mid Z(s) = s)(1 - F(s))\,ds$$

$$= \int_0^\infty E(|V'(0)| \mid T^o = -s)(1 - F(s))\,ds = \mu_1 E|V'(0)| < \infty .$$

Thus, we have proved:

Theorem 3:

Let $\{V(t), t \geq 0\}$ be a regenerative process with a measurable modification and such that $F \in G$, $\mu_1 < \infty$. Then EV_o' exists iff κ_1 exists and

$$\frac{\kappa_1}{\mu_1} = EV_o' = EV_\infty .$$

Comments:

1. Regenerative reward processes (real-valued regenerative processes) arise frequently in queuing theory. They are often of the form $V(t) = W(S(t))$, where W is a real-valued function, and $S(t)$ an abstract valued regenerative process. For example in an M/G/s queue with $\frac{\mu_G}{\lambda} < s$, the imbedded renewal sequence consists of epochs at which busy periods begin (the interarrival times satisfy $\mu_1 < \infty$, $F \in G$) $S(t)$ consists of the number of customers in service at time t with their arrival times, and the number of customers in the queue, and $W(S(t))$ may be the number of customers in service, or the number in the queue, or the unit cost of the service system for handling the number of customers present, or an indicator variable

$$W(S(t)) = \begin{cases} 1 & \text{if number in queue } = k \\ 0 & \text{otherwise} \end{cases}, \text{ etc.}$$

Assume that S is a regenerative process with arbitrary state space (F,A) and jointly measurable as a map from $(\Omega,C) \times (R,B)$ to (F,A). Here (Ω,C,P) is the probability space on which each $S(t)$ is defined, R the real line and B the Borel sets. If W is a Borel measurable real-valued function then $\{V(t) = W(S(t)), t \geq 0\}$ is a real-valued measurable regenerative process. If $\mu_1 < \infty$, $F \in G$, then since $\Pr(S(t) \in A) \to \Pr(S(\infty) \in A)$ for all $A \in A$, it follows that $\Pr(W(S(t)) \in B) \to \left(\mu_\infty W^{-1}\right)(B) = \Pr(W(S(\infty)) \in B)$, for all Borel sets. Thus, if $E|W(S(\infty))| < \infty$ then it follows from Theorem 3 that:

$$(8) \qquad \frac{1}{t} \int_0^t W(S(x))dx \to E(W(S(\infty)))$$

a.s. and in expectation. Note also that if $E\left|\frac{1}{t} \int_0^t W(S(x))dx\right|^{P_o} \to E|W(S(\infty))|^{P_o}$ for some $p_o > 1$, then $\frac{1}{t} \int_0^t W(S(x))dx \to E(W(S(\infty)))$ in L^P, for $0 \leq p \leq p_o$.

2. If $EV(t)$ converges then $EV(\infty)$ must be its limit, since $EV(\infty) = \lim_{t \to \infty} \frac{1}{t} \int_0^t E(V(s))ds$. In this case Theorem 4 is trivial. However, $E(V(t))$ may not converge and Theorem 4 may still hold. For example, start with a renewal at time 0 and let the interarrival time c.d.f. $F \in G$, have an atom at 1. Choose a regenerative process V so that $E(V(t) \mid Z(t) = 1/2) = \infty$, $E(V(t) \mid Z(T) > 0) = 0$. Then clearly $EV\left(n + \frac{1}{2}\right) = \infty$ for all integers n, but $EV(\infty) = 0$. A necessary and sufficient condition for convergence of $EV(t)$ to $EV(\infty)$ is uniform integrability of $g(s) = E(V(t) \mid Z(t) = s)$ with respect to the family $\{F_t, t \geq 0\}$, discussed in Section 1.

3. Also note that if $F \not\subset G$ but Smith's alternative conditions [5], p. 259 hold, so that $\Pr(V_t \in B) \to \Pr(V_\infty \in B)$ for all Borel sets, then Theorem 5 still applies. If $V(t)$ does not have a limiting distribution, it still holds that if $\mu_1 < \infty$, V has a measurable modification and $E|V'(0)| < \infty$, then

$$\frac{1}{t} \int_0^t V(s)ds \to \kappa_1/\mu_1 = EV'(0) \text{ , a.s. and in expectation.}$$

REFERENCES

[1] Beutler, F. J. and O.A.Z. Leneman, "The Theory of Stationary Point Processes," Acta Mathematica, pp. 159-197, (1966).

[2] Johns, M. V., Jr. and R. G. Miller, Jr., "Average Renewal Loss Rates," Annals of Mathematical Statistics, Vol. 34, pp. 396-401, (1963).

[3] Loéve, M., PROBABILITY THEORY, D. Van Nostrand, Princeton, New Jersey, (1962).

[4] Pyke, R., "Markov Renewal Processes: Definitions and Preliminary Properties," Annals of Mathematical Statistics, Vol. 32, pp. 1231-1242, (1961).

[5] Smith, W. L., "Renewal Theory and Its Ramifications," Journal of the Royal Statistical Society, Series B, Vol. 20, pp. 243-302, (1958).

[6] Smith, W. L., "Regenerative Stochastic Processes," Proceedings of the Royal Statistical Society, Series A, Vol. 232, pp. 6-31, (1955).

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
University of California, Berkeley	Unclassified
	2b. GROUP

3. REPORT TITLE

RENEWAL REWARD PROCESSES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report

5. AUTHOR(S) (First name, middle initial, last name)

Mark Brown and Sheldon M. Ross

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1969	12	6
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)	
DA-31-124-ARO-D-331	ORC 69-31	
b. PROJECT NO.		
20014501B14C		
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.	Technical Report No. 93	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
NONE	U.S. Army Research Office-Durham Box CM, Duke Station Durham, North Carolina 27706

13. ABSTRACT

SEE ABSTRACT.

DD FORM 1473 (PAGE 1)
S/N 0101-807-6811

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Renewal Reward Process						
Blackwell's Theorem for Renewal Reward Process						
Regenerative Reward Process						

DD FORM 1473 (BACK)
1 NOV 65
S/N 0101-807-6821

Unclassified
Security Classification

A-31409